Dimensional renormalizability in compactified spaces

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Abstract

We first briefly review some aspects of the techniques of dealing with ultraviolet divergences in Feynman amplitudes in an Euclidian D-dimensional space-time. Next we consider compactification of a d-dimensional ($d \leq D$) subspace. This includes effects of temperature and of compactification of $d-1$ spatial coordinates. Then we show how dimensional renormalization can be implemented for a field theory defined on this Euclidian space-time with a compactified subspace.

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I. INTRODUCTION

Studies on field theories with compactified dimensions have their theoretical roots in the finite temperature field theory historical procedure, of looking for methods paralleling temperature-independent ($T = 0$) theories, which present practical and well developed tools, as Feynman diagrams and renormalization techniques. The first systematic approach to treat a quantum field theory at finite temperature was presented in 1955 [1], the Matsubara or *imaginary-time* formalism. Since then the development of the thermal field formalism has followed the achievements of the $T = 0$ quantum field theory. The first generalization of the imaginary formalism was carried out in in 1957 [2], extending the Matsubara work to the relativistic quantum field theory, and discovering periodicity (antiperiodicity) conditions for the Green functions of boson (fermion) fields, a concept that later became known as the KMS (Kubo, Martin and Schwinger) condition.

From a topological point of view, the Matsubara formalism is equivalent to a path-integral evaluated on $\mathbb{R}^{D-1} \times \mathbb{S}^1$, where $\mathbb{S}^1$ is a circumference of length $\beta = 1/T$. As a consequence, the Matsubara prescription can be thought, in a generalized way, as a mechanism to deal with thermal effects and with spatial compactification. This concept has been developed by considering a simply or non-simply connected D-dimensional manifold with a topology of the type $\Gamma_D^d = \mathbb{R}^{D-d} \times \mathbb{S}^{1_1} \times \mathbb{S}^{1_2} \cdots \times \mathbb{S}^{1_d}$, with $\mathbb{S}^{1_1}$ corresponding to the compactification of the imaginary time and $\mathbb{S}^{1_2}, \ldots, \mathbb{S}^{1_d}$ referring to the compactification of $d-1$ spatial dimensions [3]. The topological structure of the space-time does not modify the local field equations. However, the topology implies modifications of the boundary conditions over fields and Green functions [4]. Physical manifestations of this type of topology include, for instance, the vacuum-energy fluctuations giving rise to the Casimir effect [5, 6, 7, 8, 9], or in phase transitions, the dependence of the critical temperature on the parameters of compactification [9, 10, 11, 12].

In the topology Γ_D^d, the Feynman rules are modified by introducing a generalized Matsubara prescription, performing the following multiple replacements (compactification of a d-dimensional subspace),

$$\int \frac{dk_1}{2\pi} \to \frac{1}{\beta} \sum_{n_1=-\infty}^{+\infty} , \quad \int \frac{dk_i}{2\pi} \to \frac{1}{L_i} \sum_{n_i=-\infty}^{+\infty} ; \quad k_1 \to \frac{2n_1\pi}{\beta} \quad k_i \to \frac{2n_i\pi}{L_i} , \tag{1}$$

where $L_i \,, \quad i = 2, 3..., d-1$ are the sizes of the compactified spatial dimensions.

These ideas have had recently a regain of interest, particularly as a new way to investigate the eletroweak transition and baryogenesis. For instance a recent investigation of the eletroweak phase transition has been improved in [13, 14] in the context of a 5-dimensional finite temperature theory with a compactified spatial extra dimension. These authors conclude for a first-order transition with a strength inversely proportional to the Higgs mass. Another interesting result of [13] is that up to temperatures of the order of the inverse of the compactification lentgh, reliable (low order) perturbative calculations lead to reasonable results. In particular models where the Higgs field is identified with the internal component of a gauge field in extra compactified dimensions with size of inverse TeV [15] are considered. These are known as models with gauge-Higgs unification, and are worked-out examples [16, 17, 20, 21]. Earlier references are in [18] and an overview is found in [19]. The five-dimensional (5D) case, with just one extra compactified dimension, is the simplest one and also the one which seems phenomenologically more appealing.

The situation summarized above leads to appropriate developments in field theory on spaces with compactified dimensions, in particular for implementing proper renormalization techniques in such cases. We believe that a step in this direction is considered in this paper, by setting a basis for full development of renormalization theory in space-time with spatial compactified dimensions, at zero or finite temperature.

In the following, we first make a brief overview of the fundamental aspects of renormalization theory in Sec. II, in order to make this article as self-contained as possible for a field-theorist reader. Then we show how dimensional renormalization can be implemented in an Euclidian space-time with a compactified subspace. For clear and rigorous presentations of renormalization theory in non-compactified spaces, for both commutative and non-commutative field theories, the reader is referred to [22, 23, 24].

II. GENERAL ASPECTS OF PERTURBATIVE RENORMALIZATION

For definiteness we consider the massive Euclidean $\lambda\phi^4_D$-theory described as usual, by the Lagrangian density,

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi(x)\partial^\mu \phi(x) + \frac{m^2}{2}\phi^2(x) + \frac{\lambda}{4!}\phi^4(x), \tag{2}$$

in a non-compactified Euclidian D-dimensional space-time. In this case, the Feynman amplitude for a general diagrammatic insertion G has an expression of the form (omitting vertex

factors and the overall symmetry coefficient),

$$A_G(\{p\}) \propto \int \prod_{i=1}^{I_G} \frac{d^D q_i}{(2\pi)^D} \frac{1}{q_i^2 + m^2} \prod_{v=1}^{V_G} \delta\left(\sum_{j=1}^{I} \epsilon_{vj} q_j\right), \tag{3}$$

where $\{p\}$ stands for the set of external momenta, V_G is the number of vertices, I_G is the number of internal lines and q_i is the momentum of each internal line i. The quantity ϵ_{vi} is the *incidence matrix*, which equals 1 if the line i arrives at the vertex v, -1 if it starts at v and 0 otherwise. Performing integrations over the internal momenta using the delta functions, it leads to a choice of independent loop-momenta $\{k_\alpha\}$ and we get,

$$A_G(\{p\}) = \int \prod_{\alpha=1}^{L_G} \frac{d^D k_\alpha}{(2\pi)^D} \prod_{i=1}^{I_G} \frac{1}{q_i^2(\{p\}, \{k_\alpha\}) + m^2}, \tag{4}$$

where L_G is the number of independent loops. The momentum q_i is a *linear* function of the independent internal momenta k_l and of the external momenta $\{p\}$. By power counting, we find that the integral in Eq. (4) is superficially convergent if $DL_G - 2I_G < 0$; otherwise, if $DL_G - 2I_G \geq 0$, the integral is ultraviolet divergent. So, given a diagram G, we define the quantity

$$d_G = DL_G - 2I_G \tag{5}$$

as the superficial degree of divergence of the diagram. If $d_G \geq 0$ the diagram will be ultraviolet divergent.

For any sub-diagram $S \subset G$ there are corresponding sub-integrations, and we find that if

$$d_S = DL_S - 2I_S \geq 0, \tag{6}$$

where L_S and I_S are, respectively, the number of independent loops and the number of internal lines of the sub-diagram S; an ultraviolet divergence appears associated with the sub-diagram S. Thus even if the diagram G is superficially convergent, $d_G < 0$, the Feynman integral A_G can be divergent. For this, it is enough that there is a sub-diagram S such that $d_S \geq 0$. This has been stated in Ref. [27]. A freely transposed version of this statement is:

Theorem II.1 *Let us consider a diagram G. If for all subdiagrams $S \subseteq G$ we have $d_S < 0$ the Feynman integral A_G is ultraviolet convergent. If there is at least one $S \subseteq G$, such that $d_S \geq 0$, A_G is ultraviolet divergent.*

The divergent subdiagrams of a given diagram are called *renormalization parts*. For the full renormalization process, only non-overlaping renormalization parts need to be considered [26, 27].

We present in the following an analysis in non-compactified spaces, but the general features would apply as well in the case of a compactified subspace, as it will be shown later. The basis of the perturbative renormalization method is that the starting theory is not consistent as a physical model, and this fact manifests itself as divergences. Then attempts to modify the theory are made, by introducing supplementary terms (*counterterms*) in the original Lagrangian, in such a way as to cancel the original divergences.

An important step in dimensional renormalization is dimensional *regularization*. There are different regularization methods; all of them replace divergent Feynman amplitudes with more general integrals by means of a set of supplementary parameters, such that the theory does not have ultraviolet divergences when these parameters belong to some domain. For a certain limit of these parameters we find the original theory with their divergences. This is a provisional procedure to explore more precisely the divergences to be suppressed in formal calculations. Some methods of regularization are: cutoff in the momenta, Pauli-Villars regularization, analytic regularization, lattice regularization and dimensional regularization [29, 30]. In this case the idea is to define the Feynman integrals in a generic space-time of dimension D in such a way that the divergences are recovered as poles of some functions. We will be particularly concerned with the simple integral,

$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2+M)^s} = \frac{\Gamma\left(s-\frac{D}{2}\right)}{(4\pi)^{\frac{D}{2}}\Gamma(s)} \frac{1}{M^{s-\frac{D}{2}}} \,. \tag{7}$$

We will indicate symbolically a regularized amplitude as depending on a parameter η and the suppression of the regularization as $\eta \to 0$. In what follows, unless explicitly stated, we understand *renormalized* quantities as the limit $\eta \to 0$ of the properly regularized objects.

For a renormalizable theory, we look for the organization of the set of subtractions to be performed in order to define the complete set of counterterms. When a counterterm c_{S_1} for a subdiagram S_1 with N_1 external lines is present, the modified Lagrangian contains a new vertex with N_1 lines. For any $S_2 \supset S_1$, subtracting the divergent integration corresponding to S_1 is equivalent to consider the sum

$$A_{S_2}^{\mathrm{Ren}} = A_{S_2} + c_{S_1} A_{S_2/S_1} \,,$$

where A_{S_2/S_1} is the amplitude corresponding to the reduced diagram obtained by shrinking the subdiagram S_1 to a point. If S_2 is superficially divergent (independently of the S_1-divergence), we must define another counterterm c_{S_2}, such that

$$A_{S_2}^{\mathrm{Ren}} = A_{S_2} + c_{S_1} A_{S_2/S_1} + c_{S_2}.$$

The process is continued in a recurrent manner, starting from the smallest diagram to the larger ones. It may be shown that, in order to obtain finite amplitudes, it is enough to take simultaneously all the non-overlapping subdiagrams S. This is the origin of the BPH (Bogoliubov, Parasiuk, Hepp) recursive process [26, 27, 28].

Having defined all counterterms up to a given order n, the renormalized amplitude for a diagram G of the immediatly higher order, A_G^{Ren}, is given by,

$$A_G^{\mathrm{Ren}} = \sum_{\{S\}} \left[A_{G/\{S\}} \prod_{S \in \{S\}} c_S \right] + c_G, \tag{8}$$

where c_G is present if G itself is superficially divergent. The sum in Eq. (8) is over all the families $\{S\}$ of superficially divergent non-overlapping subdiagrams of G, including the empty family. The amplitude $A_{G/\{S\}}$ corresponds to the diagram obtained by reducing to a point each subdiagram of the family $\{S\}$. In the recursive process, it is understood that the intermediary step of the regularization has been carried out, which is suppressed after the recurrence is performed up to a given order. This procedure can be generalized to take into account all renormalization parts of every diagram G. Then we can state the *Bogoliubov-Parasiuk-Hepp recurrence* [26, 28] in the form,

Theorem II.2 *We define a forest $\mathcal{F}$ of G as a set $\{S_i \subseteq G\}$ of proper (connected and 1PI) subdiagrams such that for $S_i, S_j \in \mathcal{F}$, either $S_i \subset S_j$, $S_i \supset S_j$, or $S_i \cap S_j = \emptyset$. Then the renormalized amplitude of the diagram G can be written as,*

$$A_G^{\mathrm{Ren}} = \sum_{\{S\}} \left[A_{G/\{S\}} \prod_{S \in \mathcal{F}} c_S \right] + c_G, \tag{9}$$

where c_G is present if G itself is superficially divergent.

In Eq. (9) the product of renormalization parts is to be performed following the ordering in each forest, that is from smaller to bigger diagrams. Therefore the renormalized amplitude may depend on the choice of momentum routing, that is, on the choice of the independent

loop momenta satisfying the δ-function in Eq. (3). This difficulty leads to the definition for each diagram, of sets of *admissible* momenta and for these, to the statement [27],

Theorem II.3 *The amplitude $A_G^{\mathrm{Ren}}(\{p\})$ in Eq. (9) is convergent for any diagram G in Euclidian space. Its analytical continuation to the Minkowski space defines tempered distributions.*

An essential aspect of renormalization is to determine the *renormalization parts* of the theory under consideration, that is, how many counterterms must be introduced in the theory to make it convergent. For the $\lambda\phi_D^4$ model the superficial degree of divergence is written as,

$$d_G = D - V_G(D-4) + N_G\left(1 - \frac{D}{2}\right), \tag{10}$$

where N_G is the number of external legs. For $D = 4$, $d_G \geq 0$, if, and only if, $N_G \leq 4$. This implies that to any order the only ultraviolet divergent diagrams will have $N_G = 2, 4$. From topological considerations, there are no diagrams with $N_G = 3$ in the $\lambda\phi^4$ model.

The insertions $A_{G^{(2)}}$ and $A_{G^{(4)}}$ with 2 and 4 external lines respectively, and only those, are ultraviolet divergent for $D = 4$. In this case we need to introduce only two counterterms in the theory $c^{(2)} + c^{(2)\prime}$ and $c^{(4)}$ corresponding to the diagrams with two and four external legs respectively.

The simplest case of *dimensional regularization* consists in generalizing the formula given by Eq. (7) in dimension D to a complex value D'. This may be carried out for more involved Feynman integrals, with the result that they become meromorphic functions of D', $A_G(D')$, and the ultraviolet divergences appear as poles of Gamma- functions at $D' = D$. The expansion around these poles allows us to define the *dimensional renormalization*: at each step in the Bogoliubov-Parasiuk recurrence, we perform an expansion of the dimensionally regularized amplitudes in powers of $\epsilon = D' - D$. *Dimensional renormalization* consists, essentially, in subtracting the pole terms in the limit $\epsilon \to 0$, for each renormalization part in the BPH recurrence.

The main advantage of dimensional renormalization is that, in general, it respects the symmetry properties of the theory, which are often dimensionally independent. On the contrary, in other renormalization schemes, the symmetry usually needs to be re-established by adding new finite counterterms. In practical applications dimensional renormalization must be carried out following the BPH recurrence, step-by-step. An alternative procedure

has been found within the BPHZ (Bogoliubov-Prasiuk-Hepp-Zimmermann) systematics [28], where an explicit global solution is obtained for the dimensional renormalization [35]. Other rigorous renormalization procedures are given in Refs. [34, 35, 36, 37, 38].

As far as the *definiteness of renormalization* is concerned, it is worth to recall that, whenever regularization is not suppressed, amplitudes are finite to a given perturbative order. Trouble starts when we suppress the regulator. So, let us focus on regularized objects, Feynman amplitudes, counterterms, etc..., emerging from the bare Lagrangian density (2). Two sets of counterterms, corresponding to two distinct renormalization schemes, differ by a finite counterterm. To completely define the theory it is essential to eliminate this ambiguity. This can be achieved by defining the theory with physical conditions, fixing the normalization of some Green functions at an arbitrary value of external momenta, μ. For the $\lambda\phi_4^4$ theory it is enough to fix the two- and four-point functions. The renormalized Lagrangian density is obtained from the bare Lagrangian by including counterterms,

$$\mathcal{L}^{\rm Ren} = \frac{\rm Z}{2}\partial_\mu\phi\,\partial^\mu\phi + \frac{\rm Z}{2}(m^2 + c^{(2)})\phi^2 + \frac{\rm Z^2(\lambda + c^{(4)})}{4!}\phi^4, \tag{11}$$

where $\rm Z = \sqrt{1 + c^{(2)\prime}}$. The counterterms $c^{(2)}$ and $c^{(4)}$ and Z and are dependent on the regulator η and on the arbitrary parameter μ. With the rescaling of the field, $\bar\phi = \sqrt{\rm Z}\phi$ and defining the physical mass and the renormalized coupling constant by $\bar m^2 = m^2 + c^{(2)}$ and $\bar\lambda = \lambda + c^{(4)}$ respectively, we have,

$$\mathcal{L}^{\rm Ren} = \frac{1}{2}\partial_\mu\bar\phi\,\partial^\mu\bar\phi + \frac{1}{2}\bar m^2\bar\phi^2 + \frac{\bar\lambda}{4!}\bar\phi^4. \tag{12}$$

When the regularization is suppressed, everything diverges: counterterms and, for consistency, the bare mass and coupling constant diverge, in such a way to provide $finite$ physical mass and coupling constant. The Lagrangian (12) generates perturbative series in the physical coupling constant $\bar\lambda$. The independence of physical quantities on the arbitrary mass parameter μ is expressed by the well-known Callan-Symanzik equation [31].

III. COMPACTIFICATION EFFECTS ON RENORMALIZATION

A. Compactification of imaginary time

We now address the question about the renormalizability of a theory at finite temperature. Specifically, we indicate how to use dimensional regularization and analytic Zeta-function

techniques to calculate Feynman amplitudes at $T \neq 0$. Let us start with the amplitude associated with a general diagram G having L internal loops, given by Eq. (4). Using the identity

$$\frac{1}{Q_1 \cdots Q_I} = \int_0^1 dx_1 \cdots dx_I \, \delta\left(\sum_{i=1}^{I} x_i - 1\right) \frac{(I-1)!}{[x_1 Q_1 + \cdots + x_I Q_I]^I}, \tag{13}$$

Eq. (4) can be cast in the form (from now on we suppress the subscript G from L and I)

$$\begin{aligned} A_G(\{p\}) \;=\; & \int_0^1 dx_{_1} \cdots \int_0^1 dx_{_{I-1}} \int \prod_{\alpha=1}^{L} \frac{d^D k_\alpha}{(2\pi)^D} \\ & \times \frac{(I-1)!}{[x_{_1} q_{_1}^2 + \cdots + x_{_{I-1}} q_{_{I-1}}^2 + (1 - \sum_{j=1}^{I-1} x_j) q_{_I}^2 + m^2]^I}\,, \end{aligned} \tag{14}$$

where each $q_i \equiv q_i(\{p\}, \{k_\alpha\})$ is a linear function of the loop momenta $\{k_\alpha\}$. Now, completing squares, shifting and then rescaling the integration variables, Eq. (14) can be written in the form,

$$\begin{aligned} A_G(\{p\}) \;=\; & \int_0^1 dx_{_1} \cdots \int_0^1 dx_{_{I-1}} \, f_D(\{x_j\}) \\ & \times \int \prod_{\alpha=1}^{L} \frac{d^D k_\alpha}{(2\pi)^D} \frac{(I-1)!}{[k_1^2 + \cdots + k_L^2 + \Delta^2]^I}, \end{aligned} \tag{15}$$

where $f_D(\{x_j\}) = f_D(x_1, \ldots, x_{I-1})$ and

$$\Delta^2 = \Delta^2(\{p\}, \{x_j\}; m) = g(\{x_j\})\, p^2 + m^2 \tag{16}$$

is a function of the external momenta, $\{p\}$, of the Feynman parameters, $\{x_j\}$, and of the mass m [31].

For an amplitude with L independent loops, A_G, the Matsubara prescription is applied to all k_α^0 to get the finite temperature expression,

$$\begin{aligned} A_G(\{p\}; \beta) \;=\; & \frac{1}{\beta^L} \sum_{\{l_\alpha = -\infty\}}^{\infty} \int_0^1 dx_{_1} \cdots \int_0^1 dx_{_{I-1}} f_D(\{x_j\}) \\ & \times \int \prod_{\alpha=1}^{L} \frac{d^{D-1}\mathbf{k}_\alpha}{(2\pi)^{D-1}} \frac{(I-1)!}{[\mathbf{k}_1^2 + \cdots + \mathbf{k}_L^2 + \sum_{\alpha=1}^{L} \frac{4\pi^2 l_\alpha^2}{\beta^2} + \Delta^2]^I}. \end{aligned}$$

We rewrite this equation as

$$A_G(\{p\}; \beta) = \frac{1}{\beta^L} \sum_{\{l_\alpha = -\infty\}}^{\infty} \int_0^1 dx_{_1} \cdots \int_0^1 dx_{_{I-1}} f_D(\{x_j\}) B_G(\{p\}, \{x_j\}; \{l_\alpha\}, \beta), \tag{17}$$

where

$$B_G(\{p\},\{x_j\};\{l_\alpha\},\beta) = \int \prod_{\alpha=1}^{L} \frac{d^{D-1}\mathbf{k}_\alpha}{(2\pi)^{D-1}} \frac{(I-1)!}{[\mathbf{k}_1^2 + \cdots + \mathbf{k}_L^2 + \sum_{\alpha=1}^{L} b^2 l_\alpha^2 + \Delta^2]^I}, \qquad (18)$$

with

$$b = \frac{2\pi}{\beta}.$$

To perform the integration in Eq. (18), we proceed by recurrence. We start by rewriting Eq. (18) as

$$B_G(\{p\},\{x_j\};\{l_\alpha\},\beta) = \int \prod_{\alpha=1}^{L} \frac{d^{D-1}\mathbf{k}_\alpha}{(2\pi)^{D-1}} \frac{(I-1)!}{[\mathbf{k}_1^2 + \Delta_1^2]^I},$$

with Δ_1^2 given by

$$\begin{aligned} \Delta_1^2 &= \Delta_1^2(\{p\},\{x_j\};\{l_\alpha\},m,\beta;\{\mathbf{k}_{\alpha>1}\}) \\ &= \mathbf{k}_2^2 + \cdots + \mathbf{k}_L^2 + \sum_{\alpha=1}^{L} b^2 l_\alpha^2 + \Delta^2(\{p\},\{x_j\};m). \end{aligned}$$

Then, we perform the integration over $\mathbf{k}_1$ by using the formula given in Eq. (7) and obtain

$$B_G(\{p\},\{x_j\};\{l_\alpha\},\beta) = \frac{\Gamma\left(I - \frac{D-1}{2}\right)}{(4\pi)^{\frac{D-1}{2}}} \int \prod_{\alpha=2}^{L} \frac{d^{D-1}\mathbf{k}_\alpha}{(2\pi)^{D-1}} \frac{1}{[\mathbf{k}_2^2 + \Delta_2^2]^{I-\frac{D-1}{2}}},$$

where

$$\begin{aligned} \Delta_2^2 &= \Delta_2^2(\{p\},\{x_j\};\{l_\alpha\},m,\beta;\{\mathbf{k}_{\alpha>2}\}) \\ &= \mathbf{k}_3^2 + \cdots + \mathbf{k}_L^2 + \sum_{\alpha=1}^{L} b^2 l_\alpha^2 + \Delta^2. \end{aligned}$$

The second step is to integrate over the momentum $\mathbf{k}_2$, again using Eq. (7). The result is

$$B_G(\{p\},\{x_j\};\{l_\alpha\},\beta) = \frac{\Gamma\left(I - 2\left[\frac{D-1}{2}\right]\right)}{(4\pi)^{2\left[\frac{D-1}{2}\right]}} \int \prod_{\alpha=3}^{L} \frac{d^{D-1}\mathbf{k}_\alpha}{(2\pi)^{D-1}} \frac{1}{[\mathbf{k}_3^2 + \Delta_3^2]^{I-2\left[\frac{D-1}{2}\right]}},$$

where

$$\begin{aligned} \Delta_3^2 &= \Delta_3^2(\{p\},\{x_j\};\{l_\alpha\},m,\beta;\{\mathbf{k}_{\alpha>3}\}) \\ &= \mathbf{k}_4^2 + \cdots + \mathbf{k}_L^2 + \sum_{\alpha=1}^{L} b^2 l_\alpha^2 + \Delta^2(\{p\},\{x_j\};m). \end{aligned}$$

This procedure is continued until we have integrated over all momenta. We end up with

$$B_G(\{p\},\{x_j\};\{l_\alpha\},\beta) = \frac{\Gamma\left(I - L\left[\frac{D-1}{2}\right]\right)}{(4\pi)^{L\left[\frac{D-1}{2}\right]}} \frac{1}{[\Delta_L^2]^{I-L\left[\frac{D-1}{2}\right]}}$$

where

$$\begin{aligned}\Delta_L^2 &= \Delta_L^2(\{p\},\{x_j\};\{l_\alpha\},m,\beta)\\ &= \sum_{\alpha=1}^{L} b^2 l_\alpha^2 + \Delta^2(\{p\},\{x_j\};m).\end{aligned}$$

The result for the amplitude then becomes

$$\begin{aligned}A_G(\{p\};\beta) &= \frac{1}{\beta^L}\frac{\Gamma\left(I-L\left[\frac{D-1}{2}\right]\right)}{(4\pi)^{L\left[\frac{D-1}{2}\right]}}\\ &\quad\times\int_0^1 dx_1\cdots\int_0^1 dx_{I-1} f_D(\{x_j\})\sum_{\{l_\alpha=-\infty\}}^{\infty}\frac{1}{\left[\Delta_L^2\right]^{I-L\left[\frac{D-1}{2}\right]}}\,. \end{aligned}\tag{19}$$

We recognize the sum over the set $\{l_\alpha\}$ in Eq. (19) as one of the multi-variable Epstein-Hurwitz zeta functions [32, 33] defined by,

$$Z_s^{h^2}(\nu;a_1,\ldots,a_s)=\sum_{\{n_j=-\infty\}}^{+\infty}\frac{1}{\left(\sum_{r=1}^{s}a_r^2n_r^2+h^2\right)^{\nu}}\,. \tag{20}$$

This function can be analytically continued to the whole complex ν-plane, with the result [3],

$$Z_s^{h^2}(\nu;\{a_j\})=\frac{\pi^{s/2}}{a_1\cdots a_s\,\Gamma(\nu)}\left[\Gamma\left(\nu-\frac{s}{2}\right)h^{s-2\nu}+F_s\left(\nu-\frac{s}{2};\{a_j\},h\right)\right], \tag{21}$$

where the function $F_s\left(\nu-s/2;\{a_j\},h\right)$ is the finite part, given by

$$\begin{aligned}F_s\left(\nu-\frac{s}{2};\{a_j\},h\right) &= 4\sum_{i=1}^{s}\sum_{n_i=1}^{\infty}\left(\frac{\pi n_i}{h a_i}\right)^{\nu-\frac{s}{2}}K_{\nu-\frac{s}{2}}\left(\frac{2\pi h n_i}{a_i}\right)\\ &\quad+8\sum_{i<j=1}^{s}\sum_{n_i,n_j=1}^{\infty}\left(\frac{\pi}{h}\sqrt{\frac{n_1^2}{a_1^2}+\frac{n_2^2}{a_2^2}}\right)^{\nu-\frac{s}{2}}\\ &\quad\times K_{\nu-\frac{s}{2}}\left(2\pi h\sqrt{\frac{n_1^2}{a_1^2}+\frac{n_2^2}{a_2^2}}\right)\\ &\quad+\cdots+\,2^{s+1}\sum_{\{n_i\}=1}^{\infty}\left(\frac{\pi}{h}\sqrt{\frac{n_1^2}{a_1^2}+\cdots+\frac{n_s^2}{a_s^2}}\right)^{\nu-\frac{s}{2}}\\ &\quad\times K_{\nu-\frac{s}{2}}\left(2\pi h\sqrt{\frac{n_1^2}{a_1^2}+\cdots+\frac{n_s^2}{a_s^2}}\right),\end{aligned}\tag{22}$$

and where $K_{\nu-s/2}$ denotes the modified Bessel function. The first term in Eq. (21), proportional to $\Gamma\left(\nu-s/2\right)$, has simple poles at $\nu=-n+s/2$, for $n\in\mathbb{N}$.

Taking $s = L$, $a_1 = \cdots = a_L = b = 2\pi/\beta$, $h = \Delta(\{p\},\{x_j\};m)$ and $\nu = I - L(D-1)/2$ in Eqs. (21) and (22), the L-loop amplitude, Eq. (19), becomes

$$\begin{aligned} A_G(\{p\};\beta) \;=\; & \frac{1}{2^{LD}\pi^{L(D-1)}}\left[\Gamma\left(I-\frac{LD}{2}\right)\int_0^1 dx_{_1}\cdots\int_0^1 dx_{_{I-1}}\, f_D(\{x_j\})\right. \\ & \times\frac{1}{[\Delta(\{p\},\{x_j\};m)]^{2I-LD}} \\ & +\int_0^1 dx_{_1}\cdots dx_{_{I-1}}\, f_D(\{x_j\}) \\ & \left.\times F_L\left(I-\frac{LD}{2};\{a_j=\frac{2\pi}{\beta}\},\Delta(\{p\},\{x_j\};m)\right)\right]. \end{aligned} \tag{23}$$

The first term in this expression does not depend on the temperature, $T = \beta^{-1}$, while the second term depends on the temperature in such a way that it vanishes at zero temperature, since $F_L \to 0$ as $T \to 0$ ($\beta \to \infty$). Furthermore, the first term (the $T = 0$ contribution) carries a singularity for space-time dimensions D satisfying $I - LD/2 = 0, -1, -2, \ldots$, while the temperature-dependent contribution to the amplitude, the second term, is finite. To get the renormalized amplitude, we have to suppress the singular part of the first term and add its finite part to the second, temperature-dependent, contribution. The singular part of the amplitude is easily identified by expanding the Γ-function in a Laurent series around the pole. The discussion presented so far is restricted to the compactification of the imaginary time. It equally applies to the compactification of one spatial coordinate in the Euclidian $\lambda\phi^4$ theory. The generalization of this procedure to the compactification of a subspace of dimension $d \subseteq D$ is presented in the following subsection.

B. Finite temperature and spatial compactification

The method of the previous subsection can be extended to the case where, besides imaginary time, $d-1$ spatial dimensions are also compactified. The set of compactification lengths will be denoted by $\{L_i\} = \{L_1, \cdots, L_{d-1}\}$, but no confusion arises with the number of independent loops (L) of the diagram. In this case, applying the generalized Matsubara

prescription, Eq. (1), to Eq. (15) leads to

$$\begin{aligned} A_G(\{p\};\beta,\{L_i\}) \;=\; & \frac{1}{(\beta L_1\cdots L_{d-1})^L}\sum_{\{l_{(j)\alpha}=-\infty\}}^{\infty}\int_0^1 dx_{_1}\cdots\int_0^1 dx_{_{I-1}} f_D(\{x_j\}) \\ & \times\int\prod_{\alpha=1}^{L}\frac{d^{D-d}\mathbf{k}_\alpha}{(2\pi)^{D-d}}\frac{(I-1)!}{\left[\mathbf{k}_1^2+...+\mathbf{k}_L^2+\sum_{\alpha=1}^{L}b_{j\alpha}^2 l_{(j)\alpha}^2+\Delta^2\right]^I}\,, \end{aligned}$$

where, now, $\mathbf{k}_i$ $(i=1,\ldots,L)$ are $(D-d)$-dimensional vectors and we have numbered the Matsubara frequencies with integers $l_{(j)\alpha}$ where $j=0,1,\ldots,d-1$ refer to the compactified coordinates and α has values from 1 to L, the number of independent loops of the diagram. Then the following steps are similar to those leading to Eq. (19) and we get,

$$\begin{aligned} A_G(\{p\};\beta,\{L_i\}) \;=\; & \frac{1}{(\beta L_1\cdots L_{d-1})^L}\frac{\Gamma\left(I-L\left(\frac{D-d}{2}\right)\right)}{(4\pi)^{L\left(\frac{D-d}{2}\right)}} \\ & \times\int_0^1 dx_{_1}\cdots\int_0^1 dx_{_{I-1}} f_D(\{x_j\})\sum_{\{l_{(j)\alpha}=-\infty\}}^{\infty}\frac{1}{[\Delta_{Ld}^2]^{I-L\left(\frac{D-d}{2}\right)}}, \end{aligned} \tag{24}$$

where

$$\begin{aligned} \Delta_{Ld}^2 \;&=\; \Delta_{Ld}^2(\{p\},\{x_j\};\{l_{(j)\alpha}\},m,\beta) \\ &=\; \sum_{j=0}^{d-1}\sum_{\alpha=1}^{L} b_{j\alpha}^2 l_{(j)\alpha}^2+\Delta^2(\{p\},\{x_j\};m) \end{aligned}$$

with $b_{0\alpha}=b=2\pi/\beta$ and $b_{1\alpha}=2\pi/L_1,\ldots,b_{d-1,\alpha}=2\pi/L_{d-1}$ for all $1\leq\alpha\leq L$. The sum in Eq. (24) is the multi-variable, $(d\times L)$-dimensional, Epstein-Hurwitz function,

$$Z_{dL}^{\Delta^2}\left(I-\frac{L(D-d)}{2};\{b_{0\alpha}=\frac{2\pi}{\beta}\},\{b_{j\alpha}=\frac{2\pi}{L_j}\}\right)\,,$$

which possesses an analytical extension to complex values of $\nu=I-\frac{L(D-d)}{2}$ given by Eqs. (21) and (22). Using these expressions, the regularized finite-temperature amplitude, for $(d-1)$ ccompactified spatial coordinates, is given by

$$\begin{aligned} A_G(\{p\};\beta,\{L_i\}) \;=\; & \frac{1}{2^{LD}\pi^{L(D-d)}}\left[\Gamma\left(I-\frac{LD}{2}\right)\int_0^1 dx_{_1}\cdots\int_0^1 dx_{_{I-1}} f_D(\{x_j\})\right. \\ & \times\frac{1}{[\Delta(\{p\},\{x_j\};m)]^{2I-LD}} \\ & +\int_0^1 dx_{_1}\cdots\int_0^1 dx_{_{I-1}} f_D(\{x_j\}) \\ & \left.\times\, F_{dL}\left(I-\frac{LD}{2};\{b_{0\alpha}\},\{b_{j\alpha}\},\Delta(\{p\},\{x_j\};m)\right)\right]. \end{aligned} \tag{25}$$

Again, the amplitude is separated into a zero-temperature free-space contribution ($\beta, L_i \to \infty$), which eventually has a singular part, and a contribution carrying the effects of temperature and spatial compactification, which is finite. We then state the theorem:

Theorem III.1 *Let us consider in the ϕ^4 theory, a renormalization part, a diagram $S \subseteq G$ belonging to a forest $\mathcal{F}$ of a bigger diagram G, and its related finite-temperature amplitude, with $(d-1)$ compactified spatial coordinates, $A_S(\{p\};\beta,\{L_i\})$. For the situations where $I - LD/2 = -n,\ n = 0,1,2,\ldots$, the following quantity,*

$$
\begin{aligned}
A_S^{\mathrm{ren}}(\{p\};\beta,\{L_i\}) \;=\; & \frac{1}{2^{LD}\pi^{L(D-d)}} \\
& \times\left[\frac{(-1)^n}{n!}\psi(n+1)\int_0^1 dx_{_1}\cdots\int_0^1 dx_{_{I-1}} f_D(\{x_j\})\right. \\
& \times\frac{1}{[\Delta(\{p\},\{x_j\};m)]^{2I-LD}} \\
& +\int_0^1 dx_{_1}\cdots\int_0^1 dx_{_{I-1}} f_D(\{x_j\}) \\
& \left.\times F_{dL}\left(I-\frac{LD}{2};\{b_{0\alpha}\},\{b_{j\alpha}\},\Delta(\{p\},\{x_j\};m)\right)\right],
\end{aligned}
\tag{26}
$$

where the function F_{dL} is the finite part of

$$
Z_{dL}^{\Delta^2}\left(I-\frac{L(D-d)}{2};\{b_{0\alpha}=\frac{2\pi}{\beta}\},\{b_{j\alpha}=\frac{2\pi}{L_j}\}\right)
$$

and $\psi(z) = d\ln\Gamma(z)/dz$, provides the dimensionally renormalized amplitude of the diagram S, in what superficial ultraviolet divergence in concerned. A similar statement holds for the reduced diagram G/S.

Proof: In Eq. (25) divergences occur when

$$
I-\frac{L(D-d)}{2} = -n\,,\quad n = 0,1,2,\ldots
$$

Then we use the Laurent expansion of the Γ-function around its poles,

$$
\Gamma(-n+\epsilon) = \frac{(-1)^n}{n!}\left[\frac{1}{\epsilon}+\psi(n+1)+\mathcal{O}(\epsilon)\right]
\tag{27}
$$

where $\psi(z) = d\ln\Gamma(z)/dz$, to subtract the poles of $\Gamma\left(I-\frac{L(D-d)}{2}\right)$ in Eq. (25). We are left with the finite part $\frac{(-1)^n}{n!}\psi(n+1)$. This proves the theorem. From Theorem III.1 the following theorem immediately follows.

Theorem III.2 *For all diagrams G of the ϕ^4 theory, Theorem III.1 ensures that A_G^{Ren} given by Theorem II.2 is the dimensionally renormalized Feynman amplitude of the diagram G in a space-time with a compactified subspace.*

This is easily proved since the result of theorem III.1 holds for all renormalization parts S of any diagram G. Then starting from the smallest renormalization part S, which does not contain any divergent subdiagram, the BPH recurrence in theorem II.2 ensures the dimensional renormalization.

C. Examples

We now proceed to present some examples. Consider first the one-loop amplitude shown in Fig. 1, corresponding to the first correction to the four-point function in the ϕ^4 theory. This amplitude is given by

$$\begin{aligned} A_G &= \int \frac{d^D k}{(2\pi)^D} \frac{1}{\left[(p-k)^2+m^2\right](k^2+m^2)} \\ &= \int_0^1 dx \int \frac{d^D k}{(2\pi)^D} \frac{1}{\left[k^2+x(1-x)p^2+m^2\right]^2} \,. \end{aligned} \tag{28}$$

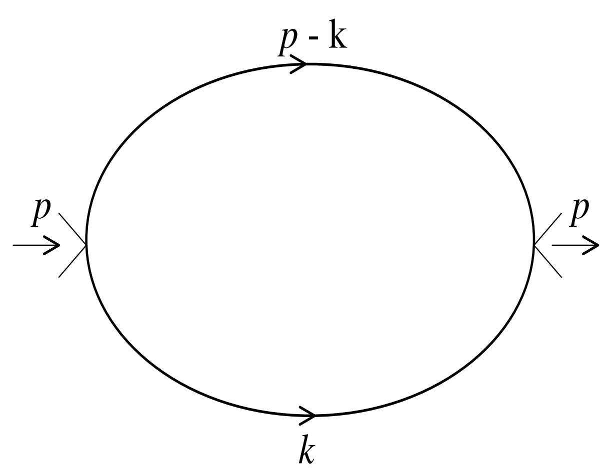

FIG. 1: One-loop contribution to the four-point function.

a) $D=4$, *with one compactified spatial dimension*:

For this one-loop case, $I - LD/2 = 0$ and the subtraction of the pole term is required; we get

$$A_G^{\text{Ren}}(p;L_1) = \frac{1}{16\pi^3}\left[-\gamma + 4\int_0^1 dx \sum_{n=1}^{\infty} K_0\left(nL_1\sqrt{x(1-x)p^2+m^2}\right)\right], \tag{29}$$

where we have used that $\psi(1) = -\gamma$, the Euler constant.

b) $D = 5$, with two compactified dimensions (β, L_1):

Taking $D = 5$ implies $I - LD/2 = -1/2$, and Eq. (25) gives directly a finite result,

$$\begin{aligned} A_G^{\text{Ren}}(p;\beta,L_1) \;=\; & \frac{1}{32\pi^3}\int_0^1 dx \,\Big[-2\sqrt{\pi}\,\Delta(p;x,m) \\ & + 4\sum_{l=1}^{\infty}\left(\frac{2\Delta}{\beta l}\right)^{\frac{1}{2}} K_{\frac{1}{2}}(l\beta\Delta) + 4\sum_{n=1}^{\infty}\left(\frac{2\Delta}{L_1 n}\right)^{\frac{1}{2}} K_{\frac{1}{2}}(nL_1\Delta) \\ & + 8\sum_{l,n=1}^{\infty}\left(\frac{2\Delta}{\sqrt{\beta^2 l^2 + L_1^2 n^2}}\right)^{\frac{1}{2}} K_{\frac{1}{2}}(\Delta\sqrt{\beta^2 l^2 + L_1^2 n^2})\Big] \end{aligned} \tag{30}$$

where

$$\Delta(p;x,m) = \sqrt{x(1-2)p^2+m^2}\,.$$

With either β or L_1 going to infinity the amplitude reduces to that with only one compactified dimension.

For a two-loop example, consider the diagram of Fig. 2 which corresponds to a second-order contribution to the propagator. In this case, we write

$$\begin{aligned} A_G \;&=\; \int \frac{d^Dk}{(2\pi)^D}\int\frac{d^Dq}{(2\pi)^D}\,\frac{1}{(k^2+m^2)[(q-k)^2+m^2][(p-q)^2+m^2]} \\ &= \int_0^1 dx\int_0^1 dy \int\frac{d^Dk}{(2\pi)^D}\int\frac{d^Dq}{(2\pi)^D}\,\frac{f_D(x,y)}{[k^2+q^2+g(x,y)p^2+m^2]^3}\,, \end{aligned} \tag{31}$$

where

$$f_D(x,y) \;=\; \frac{2}{[(x+y)(1-y)-x^2]^{D/2}}\,, \tag{32}$$

$$g(x,y) \;=\; \frac{xy(1-y)-yx^2}{(x+y)(1-y)-x^2}\,. \tag{33}$$

Taking $D = 5$, in the present case, we obtain $I - LD/2 = -2$, and so we have to subtract the pole term of the Γ-function expansion. Considering two compactified dimensions (the imaginary time, length β, and a spatial coordinate, length L_1), the renormalized amplitude

is given by

$$
\begin{aligned}
A_G^{\mathrm{Ren}}(p;\beta,L_1) \;=\; & \frac{1}{2^{10}\pi^6}\left[\frac{3-2\gamma}{4}\int_0^1 dx\int_0^1 dy f_5(x,y)\Delta^4(p;x,y,m)\right.\\
& +32\int_0^1 dx\int_0^1 dy f_5(x,y)\left\{\sum_{l=1}^{\infty}\frac{\Delta^2}{\beta^2 l^2}K_2(l\beta\Delta)\right.\\
& +\sum_{n=1}^{\infty}\frac{\Delta^2}{L_1^2 n^2}K_2(nL_1\Delta)+\sum_{l_1,l_2=1}^{\infty}\frac{\Delta^2}{\beta^2(l_1^2+l_2^2)}K_2\left(\beta\Delta\sqrt{l_1^2+l_2^2}\right)\\
& +4\sum_{l,n=1}^{\infty}\frac{\Delta^2}{\beta^2 l^2+L_1^2 n^2}K_2\left(\Delta\sqrt{\beta^2 l^2+L_1^2 n^2}\right)\\
& +\sum_{n_1,n_2=1}^{\infty}\frac{\Delta^2}{L_1^2(n_1^2+n_2^2)}K_2\left(L_1\Delta\sqrt{n_1^2+n_2^2}\right)\\
& +4\sum_{l_1,l_2,n=1}^{\infty}\frac{\Delta^2}{\beta^2(l_1^2+l_2^2)+L_1^2 n^2}K_2\left(\Delta\sqrt{\beta^2(l_1^2+l_2^2)+L_1^2 n^2}\right)\\
& +4\sum_{l,n_1,n_2=1}^{\infty}\frac{\Delta^2}{\beta^2 l^2+L_1^2(n_1^2+n_2^2)}K_2\left(\Delta\sqrt{\beta^2 l^2+L_1^2(n_1^2+n_2^2)}\right)\\
& +4\sum_{l_1,l_2,n_1,n_2=1}^{\infty}\frac{\Delta^2}{\beta^2(l_1^2+l_2^2)+L_1^2(n_1^2+n_2^2)}\\
& \left.\left.\times K_2\left(\Delta\sqrt{\beta^2(l_1^2+l_2^2)+L_1^2(n_1^2+n_2^2)}\right)\right\}\right]\,, \qquad (34)
\end{aligned}
$$

where $f_5(x,y)$ is given by Eq. (32) and $\Delta(p;x,y,m)=\sqrt{g(x,y)p^2+m^2}$, with $g(x,y)$ given by Eq. (33).

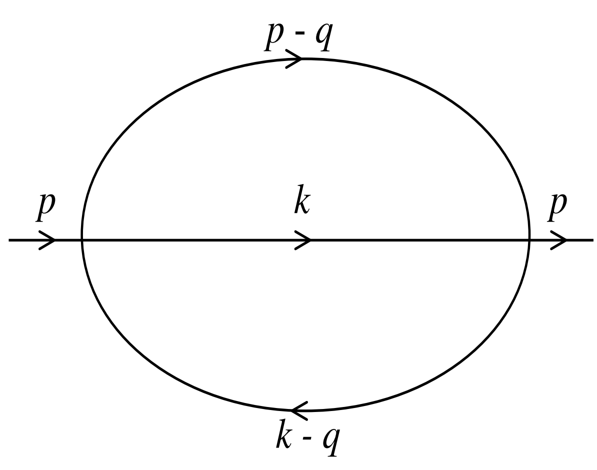

FIG. 2: Two-loop contribution to the propagator.

IV. CONCLUDING REMARKS

The results in the previous section are obtained by the concurrent use of dimensional and *zeta*-function analytic regularizations, to evaluate the integral over the continuous momenta and the summation over the generalized Matsubara frequencies corresponding to the compactified coordinates. Given a diagram G, ultraviolet divergences arise from subdiagrams $S \subseteq G$ such that the degree of divergence $d_S \geq 0$ in power-counting. These divergences appear as poles of Γ–functions with negative integer arguments (generally corresponding to even dimensions), a combination of the number of independent loops L_S, the number of internal lines I_S and of the space-time dimension D. Dimensional renormalization consists in extracting these poles, which lead to counterterms to be inserted in the BPH recurrence, given in theorem II.2, to get finite, renormalized quantities in a space-time with a compactified subspace.

From a theoretical viewpoint, the general aspects of the topic presented here can be extended to models where the matter field (bosons or fermions) is coupled with a gauge field. In these theories, an important role is played by the gauge symmetry in the discussion of perturbative renormalization. The Ward-Takahashi relations, that manifestly contain the full implications of the symmetry, have to be satisfied.

From a physical and phenomenological point of view, recently an interest in theories with extra compactified dimensions at the inverse TeV scale arose in connection with the new LHC (Large Hadron Collider) experiments. These theories provide a possible framework to throw some light on the gauge hierarchy problem [13]. Also, as we have mentioned before, a new idea brought by theories with extra dimensions is the relation between the Higgs field and the components of a gauge field. In the context of 5-dimensions [13, 14], the case of a scalar field coupled to a gauge field is considered, where the non-vanishing component of the gauge field is along the compactified dimension. Models of this type are sometimes called models with gauge-Higgs unification. Perhaps these theories provide an interesting framework for physics beyond the standard model, even though numerous problems need to be solved.

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